Solution 6

1. Identify the boundary points, interior points, interior and closure of the following sets in $\mathbb{R}$:

   (a) $[1,2) \cup (2,5) \cup \{10\}$.

   (b) $[0,1] \cap \mathbb{Q}$.

   (c) $\bigcup_{k=1}^{\infty}(1/(k+1), 1/k)$.

   (d) $\{1,2,3,\cdots\}$.

   Solution.

   (a) Boundary points: $1,2,5,10$. Interior points: $(1,2),(2,5)$. Interior: $(1,2)\cup(2,5)$. Closure: $[1,5]\cup\{10\}$.

   (b) Boundary points: All points in $[0,1]$. No interior point. Interior: the empty set ϕ. Closure: $[0,1]$

   (c) Boundary points: $\{1/k : \ k \geq 1\}, 0$. Interior points: all points in this set. Interior: This set (because it is an open set). Closure: $[0,1]$.

   (d) Boundary points $1,2,3,\cdots$. No interior points. Interior: ϕ. Closure: the set itself (it is a closed set).

2. Identify the boundary points, interior points, interior and closure of the following sets in $\mathbb{R}^2$:

   (a) $R \equiv [0,1) \times [2,3) \cup \{0\} \times (3,5)$.

   (b) $\{(x,y) : 1 < x^2 + y^2 \leq 9\}$.

   (c) $\mathbb{R}^2 \setminus \{(1,0),(1/2,0),(1/3,0),(1/4,0),\cdots\}$.

   Solution.

   (a) Boundary points: the geometric boundary of the rectangle and the segment $\{0\} \times [3,5]$. Interior points: all points inside the rectangle. Interior $(0,1)\times(3,5)$. Closure $[0,1]\times[3,5]\cup\{0\}\times[3,5]$.

   (b) Boundary points: all (x,y) satisfying $x^2+y^2=1$ or $x^2+y^2=9$. Interior points: all points satisfying $1 < x^2+y^2 < 9$. Interior $\{(x,y) : \ 1 < x^2+y^2 < 9\}$. Closure $\{(x,y) : \ 1 \leq x^2+y^2 \leq 9\}$.

   (c) Boundary points: $(0,0),(1,0),(1/2),(1/3,0),\cdots$. Interior points: all points except boundary points. Interior: $\mathbb{R}^2 \setminus \{(0,0),(1,0),(1/2),(1/3,0),\cdots\}$. Closure: $\mathbb{R}^2$.

3. Describe the closure and interior of the following sets in $C[0,1]$:

   (a) $\{f : \ f(x) > -1, \ \forall x \in [0,1]\}$.

   (b) $\{f : \ f(0) = f(1)\}$.

   Solution.

   (a) Closure: $\{f \in C[0,1] : \ f(x) \geq -1, \ \forall x \in [0,1]\}$. Interior: The set itself. It is an open set.

   (b) Closure: The set itself. It is a closed set. Interior: ϕ. Let f satisfy $f(0) = f(1)$. For every $\varepsilon > 0$, it is clear we can find some $g \in C[0,1]$ satisfying $\|g - f\|_\infty < \varepsilon$ but $g(0) \neq g(1)$. It shows that every metric ball $B_\varepsilon(f)$ must contain some functions lying outside this set.

4. Let A and B be subsets of (X, d). Show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$. Does $\overline{A \cap B} = \overline{A} \cap \overline{B}$?

   Solution. We have $\overline{A} \subset \overline{B}$ whenever $A \subset B$ right from definition. So $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$. Conversely, if $x \in \overline{A \cup B}$, $B_\varepsilon(x)$ either has non-empty intersection with A or B. So there exists $\varepsilon_j \to 0$ such that $B_{\varepsilon_j}(x)$ has nonempty intersection with A or B, so $x \in \overline{A} \cup \overline{B}$.

   On the other hand, $\overline{A \cap B} = \overline{A} \cap \overline{B}$ is not always true. For instance, consider intervals (a, b) and (b, c). We have $\overline{(a, b)} \cap \overline{(b, c)} = \{b\}$ but $\overline{(a, b) \cap (b, c)} = \phi$. Or you take A to be the set of all rationals and B all irrationals. Then $\overline{A \cap B} = \overline{\phi} = \phi$ but $\overline{A} \cap \overline{B} = \mathbb{R}$!

5. Show that $\overline{E} = \{x \in X : \ d(x, E) = 0\}$ for every non-empty $E \subset X$.

   Solution. Let $A = \{x \in X : \ d(x, E) = 0\}$. Claim that A is closed. Let $x_n \to x$ where $x_n \in A$. Recalling that $x \mapsto d(x, E)$ is continuous, so $d(x, E) = \lim_{n\to\infty} d(x_n, E) = 0$, that is, $x \in A$. We conclude that A is a closed set. As it clearly contains E, so $\overline{E} \subset A$ since the closure of E is the smallest closed set containing E. On the other hand, if $x \in A$, then $B_{1/n}(x) \cap E \neq \phi$. Picking $x_n \in B_{1/n}(x) \cap E$, we have $\{x_n\} \subset E$, $x_n \to x$, so $x \in \overline{E}$.

6. Let $E \subset (X, d)$. Show that E° is the largest open set contained in E in the sense that E° is open and $G \subset E^\circ$ whenever $G \subset E$ is open.

   Solution. Let $G \subset E$ is open. For $x \in G$, there is some $B_\varepsilon(x) \subset G$. But that means $B_\varepsilon(x) \subset E$ too, so x is an interior point of E, that is, $x \in E^\circ$. We have shown $G \subset E^\circ$. Next, we claim that E° is open. For, if x is an interior point, there is some $B_r(x) \subset E$. But then every point $y \in B_r(x)$ is also an interior point as $B_\rho(y) \subset B_r(x) \subset E$ where $\rho = r - d(x, y)$.

7. Let E be a subset of (X, d). Prove the relation $E^\circ = X \setminus \left(\overline{X \setminus E}\right)$.

   Solution. Let the right hand side be A. For $x \in A$, x does not belong to the closed set $\overline{X \setminus E}$. So there is some ball $B_\varepsilon(x)$ disjoint from $\overline{X \setminus E}$. It implies this ball is contained in E, in other words, $x \in E^\circ$. We have shown $A \subset E^\circ$. On the other hand, if $x \in E^\circ$, there is a ball $B_r(x)$ completely contained inside E. So it is disjoint from $\overline{X \setminus E}$, that is, $x \in A$. We have shown that $E^\circ \subset A$.